1a) rotational symmetries of cube

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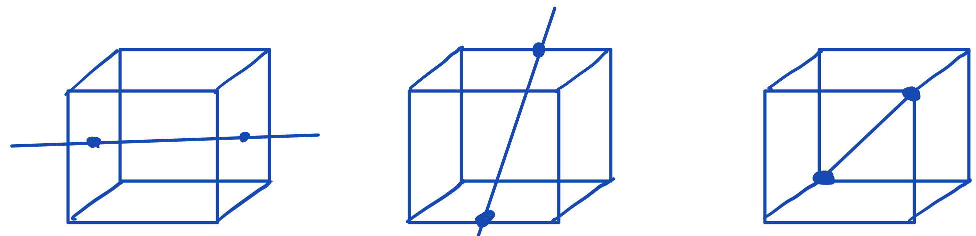

3 such axes $90^\circ, 180^\circ, 270^\circ$ — 6 such axes 180° — 4 such axes $120^\circ, 240^\circ$

$\Rightarrow 1 + 9 + 6 + 8$
$= 24 = [9]$ of S_4

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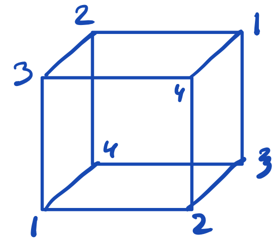
These are the permutations of the 4 diagonals:



(1423)	$(12)(34)$	(1324)	(12)	(123)

#: 6 3 6 8

1b) S_4: disjoint cycle structure :

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	#	
	1	$(e) = (1)(2)(3)(4)$
$\binom{4}{2} \rightarrow$	6	$(12) = \{(12), (13), (14), (23), (24), (34)\}$
$\binom{4}{3}\cdot 2 \rightarrow$	8	$(123) = \{(123), (124), (132), (142), (234), (243), (134), (143)\}$
$\frac{1}{2}\cdot\binom{4}{2} \rightarrow$	3	$(12)(34) = \{(12)(34), (13)(24), (14)(23)\}$
$3! \rightarrow$	6	$(1234) = \{(1234), (1324), (1423), (1243), (1342), (1432)\}$
	24	

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5 classes $\rightarrow$ 5 irreps: $\sum_i n_i^2 = 24.$

$n_1 = 1.$ $\quad n_i \leq 4.$

$n_2^2 + n_3^2 + n_4^2 + n_5^2 = 23$

n_2^2	n_3^2	n_4^2	n_5^2	
×	9	9	9	> 23
1	4	9	9	= 23 ✓
×	4	4	9	
		9	16	> 23
	4	4	16	> 23

hence

$n_1 = n_2 = 1$ 2× 1D irrep
$n_3 = 2$ 1× 2D irrep
$n_4 = n_5 = 3$ 2× 3D irrep

$1 \quad 1 \quad 4 \quad 16 \quad < 23 \quad$ etc

1c)

	(e)	(12)	(123)	$(12)(34)$	(1234)
θ	0°	180°	120°	180°	90°
$\chi^V(\theta) = 1 + 2\cos\theta$	3	-1	0	-1	1

One of these is D^V

$$\left(\frac{1}{24} \sum_g |\chi^V(g)|^2 = \frac{1}{24}(9 + 6 + 0 + 3 + 6) = 1 \right)$$

$$\langle \chi^{(1)}, \chi^V \rangle = \frac{1}{24}\left(1\cdot 3 + 6\cdot -1 + 8\cdot 0 + 3\cdot -1 + 6\cdot 1 \right) = 0$$

So no EDM allowed. (D^V irrep so no preferred direction allowed)

2a) $S_3 \cong D_3 = \mathrm{gp}\{b, c\}$ with $b^2 = c^3 = (bc)^2 = e$

$D^L(c)^3 = \mathbb{1}$, $\quad D^L(b)^2 = \mathbb{1}$, $\quad D^L(bc)^2 \stackrel{?}{=} \mathbb{1}$

$$D^L(b)\, D^L(c) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = D^L(bc)$$

$$\Rightarrow D^L(bc)^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 = \mathbb{1} \quad \checkmark \qquad \big(b\cdot c = (12)(123) = (23) \big)$$

2b) 50%

irrep : $\frac{1}{[g]} \sum_g |\chi(g)|^2 = 1 \qquad \chi(e) = \dim \text{ irrep} = d.$

$$\sum_g |\chi(g)|^2 = |\chi(e)|^2 + \sum_{g \neq e} |\chi(g)|^2 \geq d^2$$

$$\Rightarrow \frac{1}{[g]} \sum_g |\chi(g)|^2 \geq \frac{d^2}{[g]}$$

so if $d^2 > [g]$ never $= 1$. hence no irrep

50%

S_3 character table

	(e)	(c)	(b)
$D^{(1)}$	1	1	1
$D^{(2)}$	1	1	-1
$D^{(3)}$	2	-1	0

$\chi^L = (3, 0, 1)$

$\chi^L = \chi^{(1)} + \chi^{(3)}$

$D^L \sim D^{(1)} \oplus D^{(3)} \qquad \left(\text{invariant direction : } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$

2c) $\det D^L(c) = 1$, $\quad \det D^L(b) = -1$, $\quad D^L(g) \in O(3) \quad D^{L\,T}(g) D^L(g) = \mathbb{1}$

If S_3 is viewed as subgp of $O(3)$ hopfs, i.e. as $C_{3v} \cong S_3$ rather than $D_3 \cong S_3$, then $D^V \sim D^2$.

C_{3v}: $\chi^V = (3,0,1)$ $\qquad$ D_3: $\chi^V = (3,0,-1)$

3a) $O(2)$ defining rep: $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \equiv R(\theta)$

& $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$: $\{R(\theta), PR(\theta)\} \leftarrow$ defining rep of $O(2)$

3b) $\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ $\quad \vec{w} = \begin{pmatrix} w_x \\ w_y \end{pmatrix}$ $\qquad v_x w_y - v_y w_x$

$$\downarrow R(\theta)$$

$$\begin{aligned} v_x' w_y' - v_y' w_x' &= (c\theta\, v_x + s\theta\, v_y)(-s\theta\, w_x + c\theta\, w_y) \\ &\quad - (-s\theta\, v_x + c\theta\, v_y)(c\theta\, w_x + s\theta\, w_y) \\ &= c^2\theta\, v_x w_y - s^2\theta\, v_y w_x \\ &\quad + s^2\theta\, v_x w_y - c^2\theta\, v_y w_x \\ &= v_x w_y - v_y w_x \qquad \text{invariant} \end{aligned}$$

$v_x w_y - v_y w_x \xrightarrow{P} -v_x w_y + v_y w_x = -(v_x w_y - v_y w_x)$ not invariant

$\Rightarrow$ pseudoscalar in $\mathbb{R}^2$

3c) $O(2)$ invariant system $\qquad$ tensor of rank 2: T_{ij}

invariant tensor satisfies : $[D^V(g), T] = 0 \quad \forall g \in G$

2D rep D^V of $O(2)$ is an irrep, so Schur's lemma implies $T = \lambda \mathbb{1}$

hence only 1 type of invariant tensor is allowed : $T_{ij} = \lambda \delta_{ij}$

<u>weights</u>

1a)	12	2a)	10	3a)	8
1b)	12	2b)	12	3b)	10
1c)	10	2c)	8	3c)	8

grade = Σ points/10 + 1 rounded off

D. Boer, 2-4-'22